ILTA

$$\frac{\lambda_t}{1+R_t} = \beta E_t \lambda_{t+1}$$

$$E_t \lambda_{t+1} = \frac{1}{\beta P_t}$$

$$\lambda_t = \frac{1+R_t}{P_t}$$

$$z_{t,t+1} = \beta \frac{P_t}{P_{t+1}} \frac{1+R_{t+1}}{1+R_t}$$

$$\frac{1+R_t}{P_t(1+R_{zt})} = \beta E_t \frac{1}{P_{t+1}}$$

$$\frac{1}{1+R_{zt}} = \beta E_t \left[\frac{\lambda_{t+1}}{\lambda_t (1+R_{t+1})} \right] = \frac{\beta}{1+R_t} E_t \left[\frac{P_t}{P_{t+1}} \right]$$

2(TD)

$$\frac{\lambda_t}{1+R_{n,t}} = \beta E_t \frac{\lambda_{t+1}}{1+R_{n-1,t+1}}$$

$$\frac{1}{1+R_{n,t}} = \frac{\beta}{\lambda_t} E_t\left[\frac{\lambda_{t+1}}{1+R_{n-1,t+1}}\right] = \frac{\beta P_t}{1+R_t} E_t\left[\frac{\lambda_{t+1}}{1+R_{n-1,t+1}}\right]$$

$$= \frac{\beta P_t}{1+R_t} E_t\left[\frac{1+R_{t+1}}{P_{t+1}(1+R_{n-1,t+1})}\right]$$

$$= \frac{\beta P_t}{1+R_t} E_t\left[\frac{\cancel{(1+R_{t+1})}}{\cancel{P_{t+1}}} \frac{\beta \cancel{P_{t+1}}}{\cancel{1+R_{t+1}}} E_{t+1}\left[\frac{1+R_{t+2}}{P_{t+2}(1+R_{n-2,t+2})}\right]\right]$$

$$= \dots = \frac{\beta^{n-1} P_t}{1+R_t} E_t\left[E_{t+n-2}\left[\frac{\cancel{1+R_{t+n-1}}}{P_{t+n-1}\cancel{(1+R_{1,t+n-1})}}\right]\right]$$

$$= \frac{\beta^{n-1}}{1+R_t} E_t\left[\frac{P_t}{P_{t+n-1}}\right]$$

3LTD

BC time 0

$$W_0 = M_0 + \frac{B_0}{1+R_0} + E_0\left[z_{0,1} A_1\right] + \frac{B_{2,0}}{1+R_{2,0}}$$

BC time 1 multiplied by $z_{0,1}$, taking expected value as of time 0

$$E_0\left[z_{0,1}\left(B_0 + M_0 + P_0(y_0 - c_{1,0} - c_{2,0}) + A_1 + \frac{B_{2,0}}{1+R_1} - T_1\right)\right]$$

$$= E_0\left[z_{0,1} W_1\right] = E_0\left[z_{0,1}\left(\frac{B_1}{1+R_1} + M_1 + z_{1,2} A_2 + \frac{B_{2,1}}{1+R_{2,1}}\right)\right]$$

Substitute for $E_0\left[z_{0,1} A_1\right]$ from BC time 1 into BC time 0

$$W_0 = M_0\left[1 - E_0 z_{0,1}\right] + B_0\left[\frac{1}{1+R_0} - E_0 z_{0,1}\right] + E z_{0,1} T_1$$

$$+ B_{2,0}\left[\frac{1}{1+R_{2,0}} - E_0\left[\frac{z_{0,1}}{1+R_1}\right]\right] +$$

$$+ E_0\left[z_{0,1}\left(\frac{B_1}{1+R_1} + M_1 + z_{1,2} A_2 + \frac{B_{2,1}}{1+R_{2,1}}\right)\right]$$

No arbitrage requires $\frac{1}{1+R_0} = E_0 z_{0,1}$ and

$$E_0 \frac{z_{0,1}}{1+R_1} = \frac{1}{1+R_{2,0}}$$

(4/4 TD)

Keep iterating on substituting $E_t\, z_{t,t+1} A_{t+1}$, imposing no arbitrage, and take limit imposing no Ponzi, get the same PVBC as with one-period debt, except for the definition of nominal wealth

Now, assume fiscal policy is such that

$T_s = \bar{T} P_{s-1} \qquad s \neq T+1$

$T_{T+1} = P_T \left(\bar{T} + \tilde{T}_{T+1}\right)$

Realization revealed at $t<T$ and $E_s\left[\tilde{T}_{T+1}\right] = 0$ for $s<t$

For a competitive equilibrium, compute

$$\sum_{v=s}^{\infty} E_s\left[z_{s,v+1}\, T_{v+1}\right]$$

For period $s>T$, get

$$\sum_{v=s}^{\infty} E_s\left[z_{s,v+1}\, T_{v+1}\right] = \bar{T} \sum_{v=s}^{\infty} E_s\left[z_{s,v+1}\, P_v\right] =$$

$$\bar{T} \sum_{v=s}^{\infty} E_s\left[\frac{z_{s,v}\, P_v}{1+R_v}\right] = \frac{\bar{T} P_s}{1+R_s} \sum_{v=s} \beta^{v-s} = \frac{\bar{T} P_s}{(1-\beta)(1+R_s)}$$

5LTD

For period $s \in [t, T]$, get

$$\sum_{v=s}^{\infty} E_s[z_{s,v+1} T_{v+1}] = \frac{\bar{T} P_s}{(1-\beta)(1+R_s)} + E_s\left[\frac{z_{s,T}}{1+R_T} P_T \tilde{T}_{T+1}\right]$$

$$= \frac{\bar{T} P_s}{(1-\beta)(1+R_s)} + \beta^{T-s} \frac{\tilde{T}_{T+1} P_s}{1+R_s}$$

For period $s<t$

$$\sum_{v=s}^{\infty} E_s[z_{s,v+1} T_{v+1}] = \frac{\bar{T} P_s}{(1-\beta)(1+R_s)} + E_s\left[\frac{z_{s,T}}{1+R_T} P_T \tilde{T}_{T+1}\right] =$$

$$= \frac{\bar{T} P_s}{(1-\beta)(1+R_s)} + \beta^{T-s} \frac{P_s}{1+R_s} \underbrace{E_s[\tilde{T}_{T+1}]}_{=0}$$

(6LTD)

Reference steady state

Euler equation

$$1 = E_t\left[\beta(1+R_{t+1})\frac{P_t}{P_{t+1}}\right] \Rightarrow 1 = \frac{\beta(1+\bar{R})}{\bar{\pi}}$$

Asset pricing kernel

$$z_{t,t+1} = \beta\frac{P_t}{P_{t+1}}\frac{1+R_{t+1}}{1+R_t} \Rightarrow \bar{z} = \frac{\beta}{\bar{\pi}}$$

m-period interest rate

$$\frac{1}{1+R_{m,t}} = E_t\left[\frac{z_{t,t+1}}{1+R_{m-1,t+1}}\right] \Rightarrow \frac{1}{1+\bar{R}_m} = \frac{\beta}{\bar{\pi}}\frac{1}{1+\bar{R}_{m-1}} \Rightarrow 1+\bar{R}_m = \left(\frac{\bar{\pi}}{\beta}\right)^m$$

Gov't PVBC when $\frac{M_t}{P_t} \approx 0$

known B_{t-1}

$$B_{1,t-1} + \sum_{v=2}^{\infty}\frac{B_{v,t-1}}{1+R_{v-1,t}} = E_t\sum_{s=t}^{\infty} z_{ts}T_s$$

$$\Rightarrow \bar{b}P_{t-1} + \sum_{v=2}^{\infty}\phi^{v-1}\bar{b}P_{t-1}\left(\frac{\beta}{\bar{\pi}}\right)^{v-1} = \sum_{s=0}^{\infty}\left(\frac{\beta}{\bar{\pi}}\right)^{s}\bar{T}P_{t-1}\bar{\pi}^{s}$$

7CTD

Define $\hat{\phi} := \frac{\beta\phi}{\bar{\pi}}$

$$\Rightarrow \frac{\bar{b}}{1-\hat{\phi}} = \frac{\bar{T}}{1-\beta}$$

Loglinearization

$$E_t\left[\tilde{R}_{t+1} - \tilde{\pi}_{t+1}\right] = 0$$

$$\tilde{z}_{t,t+1} = \tilde{R}_{t+1} - \tilde{R}_t - \tilde{\pi}_{t+1}$$

$$\tilde{R}_{m,t} = E_t\left[-\tilde{z}_{t,t+1} + \tilde{R}_{m-1,t+1}\right] = E_t\left[-\tilde{R}_{t+1} + \tilde{R}_t + \tilde{\pi}_{t+1} + \right.$$

$$\left. - \tilde{z}_{t+1,t+2} + \tilde{R}_{m-2,t+2}\right] = E_t\left[-\tilde{R}_{t+2} + \tilde{R}_t + \tilde{\pi}_{t+1} + \tilde{\pi}_{t+2} + \tilde{R}_{m-2,t+2}\right]$$

$$= \cdots = E_t\left[-\tilde{R}_{t+m-1} + \tilde{R}_t + \sum_{s=t+1}^{t+m-1} \tilde{\pi}_s + \tilde{R}_{1,t+m-1}\right] =$$

$$= \tilde{R}_t + E_t\left[\sum_{s=t+1}^{t+m-1} \tilde{\pi}_s\right]$$

Loglinearization of BC (holding geometric maturity structure fixed)

Preliminary step $z_{t,s} T_s = \frac{\beta^{s-t} P_t}{P_s} \frac{1+R_s}{1+R_t} \left(\frac{T_s}{P_{s-1}} P_{s-1} = \right.$

(8CTD)

$$= \beta^{s-t} \frac{P_t}{P_{t-1}} \frac{P_{s-1}}{P_s} \frac{1+R_s}{1+R_t} \frac{T_s}{P_{s-1}} P_{t-1}$$

Define $b_{m,t} := \frac{B_{m,t}}{P_t}$ and divide equation by P_{t-1} on both sides prior to linearization

$$\tilde{b}_{1,t-1} \bar{b} + \sum_{v=2}^{\infty} \phi^{v-1} \tilde{b}_{1,t-1} \bar{b} \left(\frac{\beta}{\bar{\pi}}\right)^{v-1} -$$

$$- \sum_{v=2}^{\infty} \phi^{v-1} \bar{b} \tilde{R}_{v-1,t} \left(\frac{\beta}{\bar{\pi}}\right)^{v-1} = E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} \bar{T} \left(\tilde{\pi}_t - \tilde{\pi}_s + \tilde{R}_s - \tilde{R}_t + \tilde{T}_s\right)\right]$$

Divide LHS by $\frac{\bar{b}}{1-\hat{\phi}}$ and RHS by $\frac{\bar{T}}{1-\beta}$, use $E_t[\tilde{R}_{t+s} - \tilde{\pi}_{t+s}] = 0 \; s>0$

$$\tilde{b}_{1,t-1} - (1-\hat{\phi}) \sum_{v=2}^{\infty} \hat{\phi}^{v-1} \left[\tilde{R}_t + E_t\left[\sum_{s=1}^{v-1} \tilde{\pi}_{t+s}\right]\right] =$$

$$= (1-\beta) E_t \sum_{s=t}^{\infty} \beta^{s-t} \tilde{T}_s - \beta(\tilde{R}_t - \tilde{\pi}_t)$$

$$\Rightarrow \tilde{b}_{1,t-1} + (\beta-\hat{\phi}) \tilde{R}_t - (1-\hat{\phi}) \sum_{s=1}^{\infty} E_t \tilde{\pi}_{t+s} \left(\sum_{v=2}^{\infty} \hat{\phi}_{v-1}\right) = (1-\beta) \sum_{s=t}^{\infty} \beta^{s-t} E_t \tilde{T}_s + \beta \tilde{\pi}_t$$

$$\tilde{b}_{1,t-1} + (\beta-\hat{\phi}) \tilde{R}_t = (1-\beta) \sum_{s=t}^{\infty} \beta^{s-t} E_t \tilde{T}_s + (1-\hat{\phi}) \sum_{s=1}^{\infty} \hat{\phi}^{s} E_t \tilde{\pi}_{t+s} + \beta \tilde{\pi}_t$$