Negative Nominal Interest Rates *

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The determination of inflation is one of many examples in which economic outcomes are driven by an intricate interaction between private expectations and government policy. In these instances, achieving a good equilibrium outcome (e.g., low and stable inflation) requires the policymaker to adopt rules that are not only compatible with the desired outcome, but that also avoid the existence of different equilibria: it is a problem of *strict implementation*. Recently, some solutions to the implementation problem have generated a heated debate, spurred by a surprising disagreement on setting apart equilibrium conditions from restrictions on government policy that must hold under all contingencies. An example of this is the controversy over the fiscal theory of the price level.[1]

In this paper, we consider an even more paradoxical case, namely, the zero bound on nominal interest rates. While most people view the bound to be a constraint on monetary policy, which cannot be violated under any contingency, the traditional macroeconomic models, based on a notion of competitive equilibrium adapted for the presence of a big player, make it equally possible to interpret the zero bound as an equilibrium condition.[2] Indeed, negative nominal

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[1] See e.g., Woodford [10], Sims [9], Kocherlakota and Phelan [4], Bassetto [2].

[2] Schmitt-Grohé and Uribe [7] study a coherent model based on the assumption that it is possible for the

rates are deemed impossible in equilibrium because they afford an arbitrage, and the absence of arbitrage is normally considered an equilibrium condition that is required for market clearing.

In macroeconomic models with a large number of private individuals and no government, the notion of competitive equilibrium draws a clear and simple distinction between individual constraints and aggregate consistency conditions, such as market clearing: the former should be met at all conceivable prices, while the latter are only met at the equilibrium price(s). It is not specified how prices form out of the actions of the players, nor is it necessary to do so. The simple distinction is lost with the introduction of a large player. When a large player is present, two types of aggregate restrictions arise: those that constrain the large player under all contingencies, and those that describe imbalances that will be corrected by the adjustment of prices to their equilibrium level. The standard model fails to spell out an unambiguous distinction between the two, even in the stark case of negative nominal rates. As we show below, the distinction is very important in obtaining results about uniqueness of an equilibrium; this suggests that a proper analysis requires a richer model.

In section 1, we display a simple, standard, cash-in-advance model in which the ability of the government to commit to negative nominal rates is important in achieving uniqueness of an equilibrium. In section 2, we develop a richer model that explicitly accounts for the way prices form out of the actions of the private sector and the government/central bank.[3] Within that model, we are able to precisely ascertain what actions are available to the central bank. We show that, under suitable assumptions, a commitment to negative nominal rates is equivalent to a 100% tax on nominal wealth, and that the anticipation of negative nominal rates is self-defeating. Within this model, a commitment to negative nominal rates is therefore a valid way of preventing deflationary equilibria. Section 3 concludes.

government to commit to negative nominal interest rates under some contingencies.

[3]Throughout the paper, the government and the central bank are treated as a single entity, so the two terms should be viewed as interchangeable.

1 A Cash-in-Advance Economy

Consider an economy populated by a continuum of identical private agents and a central bank. The preferences of the private agents are given by

$$\sum_{t=0}^{\infty} \beta^t [2\sqrt{c_{1t}} + 2\psi\sqrt{c_{2t}} - \eta(Y_{1t} + Y_{2t})], \tag{1}$$

where c_{1t} is the consumption of "cash goods," c_{2t} is the consumption of "credit goods," and Y_{it} is output of cash ($i = 1$) or credit ($i = 2$) goods, which is produced through labor alone at a one-to-one constant rate. Cash goods can only be purchased using cash issued by the central bank. There is no uncertainty, which implies that we also rule out sunspot equilibria, and we only consider symmetric equilibria, in which all households have the same level of wealth at the end of the period.

At each time t, a household can purchase three assets: cash ($M_t \geq 0$), interest-bearing bonds (B_{t+1}) and reserves held at the central bank ($X_{t+1} \geq 0$), that carry no interest.

Households maximize (1) subject to the following constraints:[4]

$$M_t + X_{t+1} + \frac{B_{t+1}}{R_t} \leq W_t + H_t, \tag{2}$$

$$P_t c_{1t} \leq M_t, \tag{3}$$

$$W_{t+1} = M_t + P_t(Y_{1t} + Y_{2t} - c_{1t} - c_{2t}) + B_{t+1} + X_{t+1}, \tag{4}$$

and

$$W_{t+1} \geq 0, \tag{5}$$

where R_t is the one-period nominal interest rate, W_t is nominal wealth at the beginning of period t, H_t are nominal transfers from the government (taxes, if negative), and P_t is the price level in period t.[5]

[4]Equation (5) effectively prevents households from borrowing. This could be relaxed with no consequence on the results, at the price of complicating the model.

[5]We already imposed that the price of cash and credit goods is the same. We will formally prove later that this has to be the case in equilibrium.

We assume that the central bank uses the interest rate as its policy instrument, and follows a Taylor rule:

$$R_{t+1} = \bar{\pi}/\beta + F(\pi_t - \bar{\pi}). \tag{6}$$

$\bar{\pi}$ is the central bank's inflation target, F is an increasing function with $F(0) = 0$, so that the nominal interest rate responds positively to previous deviations from intended inflation, with $\pi_t = P_t/P_{t-1}$.

To complete the model, a fiscal policy rule is needed. We choose a rule that does not rely on the fiscal theory of the price level to completely rule out indeterminacy. We assume that the government sets nominal lump-sum taxes H_t in order to attain an exogenous and constant level $\bar{B}$ for the aggregate ratio $(W_t + H_t)/P_{t-1}$. Furthermore, in each period t, the government levies a real lump-sum tax ϵ of the credit good that it uses to buy back nominal claims. We think of ϵ as a very small number. We rule out equilibria with explosive inflation; Obstfeld and Rogoff [6] discuss ways to do so.

When the constraint (5) is not binding, the first-order conditions for household optimization imply that $c_{1t} = (\eta R_t)^{-2}$, $c_{2t} = (\psi/\eta)^2$, that the prices of cash and credit goods must be the same, and that $\pi_t = \beta R_t$. Combining these equations with (6), the entire equilibrium can be characterized starting from a single difference equation, which describes the behavior of the price level, together with the (exogenous) initial condition P_{t-1}:

$$\pi_{t+1} = \bar{\pi} + \beta F(\pi_t - \bar{\pi}). \tag{7}$$

Appendix A discusses possible cases in which the constraint (5) binds; in these cases, a new difference equation, which quickly leads to hyperinflation, applies.

When $\beta F'(.) > 1$ everywhere, (7) implies that price changes will have an explosive path, unless inflation is always at the steady state $\bar{\pi}$. A unique nonexplosive equilibrium ensues, with inflation stable at the central bank target.

However, it is not possible to choose the function F so that $\beta F'(.) > 1$ everywhere without violating the zero bound on nominal interest rates. As shown in Benhabib, Schmitt-Grohé and Uribe [3] for a similar model, imposing a zero bound necessarily creates a stable steady state, which means that a continuum of equilibria emerge and that the price level is indeterminate.

Within this model, the ability of committing to negative nominal interest rates is therefore crucial in driving the determinacy of the price level. However, the model itself offers very little guidance on whether it is sensible to assume that nominal interest rates can be negative. In a competitive environment without a government, $R_t \geq 1$ is not a constraint on any individual decision; rather, it is an equilibrium condition that must be satisfied in order to ensure that no arbitrage is present. It is the "Walrasian auctioneer," the black box that leads the economy to an equilibrium, that will ensure that this bound is not violated.

When the government is present, and it uses R_t as its policy instrument, it can plausibly be argued that it should be bound not to offer private households an arbitrage, in the same way as the "Walrasian auctioneer" cannot name prices (such as $R_t < 1$) that lead to an arbitrage. However, the central bank sets $R_t < 1$ only if inflation drops sufficiently low. It can also be coherently argued that the anticipation of an arbitrage would force immediate price adjustments that would steer the economy out of the "deflation" region and back towards the steady state.

2 A Full Description of the Government Strategy

In this section, we describe an economy very similar to the one above, but where the timing of moves, the possible actions of each player and the price formation mechanism are made explicit.

The following sequence of events unfolds within each period. First, the central bank sets the nominal interest rate, according to (6). More specifically, the central bank takes the following commitments:

- all holders of maturing bonds, reserves or cash are entitled to receive new cash or reserves at par;
- all holders of maturing bonds, reserves or cash can exchange them for bonds maturing in the next period at the nominal interest rate R_t. Up to the borrowing limit, private agents can also take the reverse position in the trade, borrowing at the same rate to purchase cash or reserves.

- We do not impose any upper limit to balances held against the government/central bank; households can even have an infinite balance in their account. Also, bank notes can be issued in any denomination, including infinite. However, the central bank is subject to a technological constraint that limits to N the maximum number of bank notes per capita that it can print within each period. If the demand exceeds N, the central bank is forced to ration. Bank notes are perfectly durable.

After trading assets with the central bank, the timing of the economy is very similar to Lagos and Wright [5]. Households separate and head to a specialized market, where goods are produced and exchanged in anonymous, pairwise matches. Cash is the only available means of payment, as bonds and reserves only exist as an electronic record at the central bank, which cannot be accessed during the match. Each household faces a probability $\sigma \leq 1/2$ of meeting somebody that can produce its desired cash good, and there are no meetings in which both households can produce for each other. To keep the economy close to the cash-in-advance economy of the previous section, we assume that prospective buyers make take-it-or-leave-it trade offers.[6]

Once specialized trade has occurred, households gather again in a centralized market where the "credit good" is traded. Trade takes place as in Shubik [8]: prospective buyers submit unconditional bids in any combination of cash, reserves or bonds maturing next period; prospective sellers submit unconditional bids for the goods they wish to sell; and trade takes place at the price given by the ratio of the bids. The government coerces households to work to produce an amount ϵ of the centralized good, which it sells on this market. With a continuum of households, no individual household can affect the price, hence the competitive assumption of price-taking behavior applies.[7] The price on the centralized market is what the government/central bank observes and reacts to. After trade and consumption of the credit good have taken place, the government levies a nominal tax $-H_{t+1}$ to attain the exogenous and constant level $\bar{B}$ for the ratio $(W_{t+1}+H_{t+1})/P_t$.[8] This tax is applied directly to the households' accounts of bonds and/or

[6]This ensures that the price of cash and credit goods is the same when cash and maturing bonds trade at par.

[7]If any infinite bids are placed in either of these Shapley-Shubik markets, the goods are equally divided among all households that submitted infinite bids.

[8]A household that cannot meet this obligation without attaining a negative value of nominal wealth at the

reserves maturing next period. The central bank also sets $R_{t+1} = \bar{\pi}/\beta + F(\pi_t - \bar{\pi})$, as in the previous section.

Household preferences are still given by (1).

Having fully described the strategy the government adopts, we are left with an anonymous game played among a continuum of households. We define an equilibrium as a Nash equilibrium of the game above.[9] We still restrict ourselves to equilibria with symmetric outcomes, in which all households share the same balance at the end of each period. We also only consider equilibria in which bank notes trade at their face value. In this case, our economy is almost identical to the cash-in-advance economy of the previous section.

We sketch the key features of an equilibrium, while appendix B contains a more formal treatment. First, consider a potential seller in the specialized market. The disutility from producing either good is linear and the same; furthermore, cash, reserves and bonds are perfect substitutes as of the beginning of the next period, when households can reshuffle their portfolio. The seller will thus be indifferent between producing for the buyer in the specialized market and producing in the centralized market, provided the price of the two transactions is the same; since we assumed the buyer makes it a take-it-or-leave-it offer, she chooses the quantity to buy, at the same price at which the credit good is anticipated to trade in the centralized market;[10] this is identical to the economy of the previous section. Contrary to the previous section, only some households trade in the cash good in each period, and of those, some are sellers only and others are buyers only. However, linearity implies that it is an optimal choice[11] for households to compensate their different holdings of wealth by producing more or less for the centralized market, so that all of them start each period with the same nominal wealth.

While nominal rates are nonnegative, an equilibrium process for prices is described by (7),

beginning of period $t+1$ is punished as in Bassetto [2].

[9]Bassetto [1] discusses differences between Nash and sequential equilibria for our environment. The difference is unimportant for our purposes.

[10]We must assume that preferences are such that buyers will not demand the specialized good up to the point that would make it optimal for a seller to produce a negative amount in the centralized market.

[11]This is *an* optimal choice, not *the* optimal choice: since the disutility from supplying output in *all* periods is linear, nonsymmetric equilibria could emerge.

as before.[12] Equilibrium determinacy hinges again on the stability properties of this difference equation. However, we are now in a position to consider the consequences of paths along which the central bank sets interest rates to a negative number. Suppose $R_t \geq 1$ for all periods up to period T, and $R_T < 1$. The household budget constraint, which is still (2), allows a household to accumulate infinite balances for period $T+1$ by borrowing in bonds and accumulating cash and reserves with the central bank. Infinity acts in exactly the same way as an upper bound on the *stocks* of nominal claims ($B_{T+1} + X_{T+1}$ and M_T, respectively): all households will attain this upper bound, independently of their level of wealth at the beginning of period T. Negative nominal rates at T imply thus that all balances remaining after the end of period $T-1$ are worthless. This means that no bids for selling output of the credit good will be placed in period $T-1$; hence, all bond balances, cash and reserves will be used to bid against the exogenous amount ϵ provided by the government. When ϵ is sufficiently small (or $\bar{B}$ sufficiently large), the resulting inflation will be very high. However, if inflation is above target in period $T-1$, then the Taylor rule would not call for negative nominal rates in period T: this proves that there can be no equilibria with $R_T < 1$.

According to the reasoning above, the central bank can indeed commit to a Taylor rule that involves negative nominal interest rates, and the forces of equilibrium will ensure price level determinacy. An explicit account of the interaction between households and the central bank is essential in deriving these results, as it highlights three key ingredients:

(i) When setting negative nominal interest rates, the central bank commits to deliver an infinite amount of nominal claims, rather than goods. While it is impossible for a government to deliver an infinite amount of goods, it is perfectly conceivable to state that a nominal balance is "infinite."

(ii) In the environment above, an episode of negative nominal interest rates act as a 100% tax on nominal wealth. This is why households try to get rid of nominal balances if they ever anticipate such an episode.

[12]The analysis of the case in which (5) binds yields again explosive paths, as for the previous section, and is detailed in appendix B.

(iii) The central bank commits to negative nominal rates if inflation drops sufficiently low, but the actions that households would optimally take in anticipation of such an event would generate high inflation instead. This contradiction is at the root of the central bank's success in uniquely implementing the desired equilibrium.

3 Conclusion

In this paper, we have laid out a complete characterization of an economy where it is clear how the central bank can commit to negative nominal interest rates, and that this ability can be used to strictly implement a given equilibrium.

Two remarks on the scope of this result are in order:

(i) While the strategy of committing to negative rates in response to undesirably low inflation was here successful at preventing it, this is specific to the application at hand. Simply committing to negative nominal rates in response to a potentially unwanted equilibrium in a generic model would not necessarily be a successful strategy; rather, it may generate new equilibria, where the economy does take the path to negative rates and to the disastrous destabilization of the monetary system that follows.

(ii) The implementation strategy designed here relies on steering the economy towards extremely undesirable outcomes in some circumstances, albeit those circumstances will never arise if the model is correct. In terms of actual policy advice, this is a worrisome feature. If the model is misspecified, the economy may still take a path that is supposedly "impossible" under the computed equilibrium, with severe consequences. Furthermore, it may be very hard to set up institutions or procedures that ensure adherence to a committed rule, when the benefits of deviating ex post are extremely large.

More in general, our exercise is an illustration of a fruitful technique for studying implementation problems in macroeconomics. In designing fiscal and monetary policy rules that ensure uniqueness of an equilibrium, more attention should be devoted to opening the black box of competitive equilibrium. This requires placing far greater emphasis on the specific way households, firms and the government interact through markets, and policy design involves a full description of the actions the government can take at each time and in each circumstance.

A Competitive Equilibrium Characterization

In this appendix, we characterize in detail the equilibrium conditions for the problem of section 1.

First, consider the household problem. Let $\beta^t \lambda_t$, $\beta^t \mu_t$, $\beta^{t+1}\phi_{t+1}$, and $\beta^{t+1}\rho_{t+1}$ be the Lagrange multipliers on (2), (3), (4), and (5), respectively.

Assuming $R_t \geq 1$,[13] the household first-order conditions are

$$
\begin{aligned}
\frac{1}{\sqrt{c_{1t}}} &= P_t(\beta\phi_{t+1} + \mu_t) \\
\frac{\psi}{\sqrt{c_{2t}}} &= \beta\phi_{t+1} P_t \\
\eta &= \beta\phi_{t+1} P_t \\
\lambda_t &= \mu_t + \beta\phi_{t+1} \\
\frac{\lambda_t}{R_t} &= \beta\phi_{t+1} \\
X_{t+1} &= 0 \text{ if } R_t > 1 \\
\phi_{t+1} &= \lambda_{t+1} + \rho_{t+1}
\end{aligned}
\tag{8}
$$

These equations must hold for all $t \geq 0$. Notice that (8) is the first-order condition both for Y_{1t} and Y_{2t}. We used this fact to establish that the price of cash and credit goods must be the same, as long as (voluntary) production of both goods is strictly positive.[14] The Inada condition, together with $\epsilon < (\psi/\eta)^2$, ensures that this must be the case in any equilibrium.

First, consider the case in which (5) is never binding. In this case, the first-order conditions are met by the values provided in the main text. In order to complete the proof that this

[13]When $R_t < 1$ at any time t, the household problem does not have a bounded solution.

[14]Notice that Y_{2t} is voluntary production, which excludes the amount ϵ worked for the government.

is the solution, we also need to verify that, in the described equilibrium, the present value of government transfers is finite and that the household transversality condition holds.

For $t \geq 1$, we have

$$\begin{aligned} H_t =& \bar{B}P_{t-1} - W_t = \bar{B}P_{t-1} - M_{t-1}(1 - R_{t-1}) + P_{t-1}\epsilon - R_{t-1}(W_{t-1} + H_{t-1}) = \\ & \bar{B}P_{t-1} - P_{t-1}(\eta R_{t-1})^{-2}(1 - R_{t-1}) + P_{t-1}\epsilon - R_{t-1}\bar{B}P_{t-2} = \\ & P_{t-1}\left[-\left(\frac{\beta}{\eta}\right)^2 \pi_{t-1}^{-2}\left(1 - \frac{\pi_{t-1}}{\beta}\right) + \epsilon + \bar{B}\left(1 - \frac{1}{\beta}\right)\right] \end{aligned}$$

Hence,

$$\sum_{t=1}^{\infty} \frac{H_s}{\prod_{s=0}^{t-1} R_s} = P_{-1}\sum_{t=1}^{\infty}\beta^s\left[-\left(\frac{\beta}{\eta}\right)^2 \pi_{t-1}^{-2}\left(1 - \frac{\pi_{t-1}}{\beta}\right) + \epsilon + \bar{B}\left(1 - \frac{1}{\beta}\right)\right]$$

When $\pi_{t-1} \in [\beta, +\infty)$, the argument inside brackets is bounded, which ensures that the sum converges.

The transversality condition requires

$$\lim_{t\to\infty} \frac{W_t}{\prod_{s=0}^{t-1} R_s} = 0 \tag{9}$$

We have

$$W_t = M_{t-1}(1 - R_{t-1}) - P_{t-1}\epsilon + \bar{B}P_{t-2}R_{t-1} = P_{t-1}\left[\left(\frac{\beta}{\eta}\right)^2 \pi_{t-1}^{-2}(1 - \pi_{t-1}) - \epsilon + \frac{\bar{B}}{\beta}\right] \tag{10}$$

Hence,

$$\frac{W_t}{\prod_{s=0}^{t-1} R_s} = \beta^t P_{-1}\left[\left(\frac{\beta}{\eta}\right)^2 \pi_{t-1}^{-2}(1 - \pi_{t-1}) - \epsilon + \frac{\bar{B}}{\beta}\right]$$

which implies that (9) holds. Equation (10) also shows that in this equilibrium $W_t > 0$, at least if $\bar{B}$ is sufficiently large.

Next, we consider the possibility of equilibria in which the constraint (5) is binding. Suppose that $\rho_{t+1} > 0$. Manipulating the first-order conditions, we can still obtain $c_{1t} = (\eta R_t)^{-2}$, $c_{2t} = (\psi/\eta)^2$, and the equality of the price of cash and credit goods. We now obtain $\pi_{t+1} > \beta R_{t+1}$, so the difference equation in the main text no longer applies. In order for $\rho_{t+1} > 0$ to happen in equilibrium, we must have $W_{t+1} = 0$, i.e.,

$$P_{t-1}\left[R_t\bar{B} + \pi_t(\eta R_t)^{-2}(1 - R_t) - \epsilon\pi_t\right] = 0 \tag{11}$$

If $\rho_t = 0$, we can substitute $R_t = \pi_t/\beta$ and obtain a quadratic equation in π_t:

$$\pi_t^2 \left(\frac{\bar{B}}{\beta} - \epsilon\right) - \beta\eta^{-2}\pi_t + \beta^2\eta^{-2} = 0$$

This equation has no solution when

$$\beta^2\eta^{-2} \left[\eta^{-2} - 4\left(\frac{\bar{B}}{\beta}\right) - \epsilon\right] < 0 \tag{12}$$

which is guaranteed to happen if $\bar{B}$ is sufficiently large. As an example, when ϵ is small, this condition is abundantly met when $\bar{B}$ to be equal to η^{-2}, the approximate amount of real money balances held by households when nominal interest rates and inflation are low.[15].

In this case, ρ_{t+1} can only happen if $\rho_t > 0$ as well: the resulting equilibria must have the property that (5) is binding from $t = 0$ up to some time T (possibly infinite), and never thereafter. Once the constraint stops binding, inflation is driven by the difference equation (7). While $\rho_t > 0$, the behavior of inflation is characterized by a difference equation that can be derived substituting (6) into (11). We want to establish conditions under which this difference equation is explosive, under the assumption that $\beta F'(.) > 1$. Working through some tedious algebra, it can be showed that π_t is a strictly convex function of R_t, and it reaches a minimum at

$$\frac{-1 + \sqrt{1 + 3\epsilon\eta^2}}{\epsilon\eta^2} \approx 3/2$$

with a value of

$$\pi_{\min} \equiv \frac{B(-1 + \sqrt{1 + 3\epsilon\eta^2})^3}{\epsilon^2\eta^2(1 + 2\epsilon\eta^2 - \sqrt{1 + 3\epsilon\eta^2})} \approx \frac{27\bar{B}\eta^2}{4}$$

Even under the conservative value of $B = \eta^{-2}$, inflation must be above 600% for (5) to bind.

We want to show that, when $\beta F'(.) > 1$, this new difference equation is also explosive. To do so, we proceed as follows. Our difference equation is of the form $\pi_t = \Phi(\bar{\pi}/\beta + F(\pi_{t-1} - \bar{\pi}); \bar{B})$. We explicitly leave $\bar{B}$ as a parameter; we will need to study its interaction with π_{t-1} to decide what is a "conservative value" for it down the road. We assume that $\bar{\pi}/\beta$ is less than $\pi_{\min}$, which is quite reasonable, given the value found above. Then, if $\beta F'(.) > 1$, it follows that

[15] If $\bar{B}$ drops below this level, households would need to borrow from the government in the debt market to meet their money demand when interest rates and inflation are low.

$\Phi_1(\bar{\pi}/\beta + F(\pi_{t-1} - \bar{\pi}); \bar{B}) > \Phi_1(\pi_{t-1}; \bar{B})$.[16] Convexity implies that $\Phi_1(\pi_{t-1}; \bar{B}) \geq \Phi_1(\pi_{\min}; \bar{B})$. Next, notice that $\pi_{\min}$ is (positively) proportional to $\bar{B}$. Φ_1 is also proportional to $\bar{B}$, and we will show that it is positive at $\pi_{\min}$ even under the conservative estimate of $\pi_{\min}$. It follows that $\Phi_1(\pi_{\min}; \bar{B})$ is increasing in $\bar{B}$, so we can get a lower bound by choosing a conservatively low value of $\bar{B}$. If we choose $\bar{B} = \eta^{-2}$ as before, we obtain $\Phi_1(\pi_{\min}; \bar{B}) \approx 14.5$. Any number bigger than 1 implies that we are in the explosive region, and convexity rules out the possibility of a comeback. It thus follows that inflation will certainly grow explosively after at most two periods. Any trick that monetary and/or fiscal authorities are able to play to rule out explosive inflation paths would thus rule out equilibria in which (5) is binding.

B Nash Equilibrium Characterization

B.1 Description of the Game

The main text contains a complete description of the game, with the exception of the rationing rule in the case in which more than N bank notes per capita are requested. This rule is unimportant for our purposes; we will show below that households will need only 1 bank note per capita in the equilibria we are interested in. For the sake of completeness, we assume that, if total demand for bank notes exceeds N per capita, the central bank sets an upper bound $\bar{N}$ and offers each household the minimum between $\bar{N}$ and the amount of notes requested. $\bar{N}$ is set so that total supply of bank notes is exactly N.[17]

Public and private histories for the game are defined recursively. At time 0, households start trading with the central bank with both the private and public histories being null. They then move to the specialized market; the public history contains the distribution of money (number and denomination of bank notes), debt and reserves in the population; in addition to this, a household's private history contains its own amount of money balances, bonds and reserves, the type of household it is paired with (buyer, seller or bad match) and the amount of money

[16] Φ_1 is the derivative with respect to the first argument.

[17] If all households demand more than N notes, it follows that $\bar{N} = N$.

that this potential trading partner brings to the match.[18] When households move back to the centralized market, the post-trade distribution of money holdings becomes part of the public history; if the household was matched with a buyer or a seller, its private history is augmented by the offer made by the buyer and the acceptance/rejection decision of the seller. Finally, the public history at the beginning of the period 1 includes the aggregate distribution of period-0 bids in the centralized market, and the private history also includes a household's own bid on the market. Thereafter, histories keep growing adding the elements described above.

A household strategy is a mapping from private histories into feasible actions. By its choices, a household cannot affect the price at which the centralized good is traded, nor the sequence of government taxes/transfers. In the specialized market, a household can affect the terms at which it trades with their partner in each period, but it cannot affect the terms at which all other matched households trade. Since each household trades with a measure 0 set of other households throughout the entire history of the game, the actions that a household takes in the current period in the specialized market cannot affect public histories, nor can they affect the behavior of almost all other households, with whom it will never be matched.[19] A strategy profile is a distribution of strategies across households. We only study strategy profiles in which all households take the same strategy (but not the same actions!).

B.2 Equilibrium Conditions

Now, consider a household that walks into the centralized market in period t.[20] Given the current public history and the strategy profile of other households, the household is able to compute the entire sequence of prices that will prevail in the centralized market, as well as transfers and

[18]The ex ante probability distribution over matches gives a household a probability σ of meeting a buyer, a probability σ of meeting a seller, and a probability $1 - 2\sigma$ of having a bad match; the ex ante distribution of money holdings of the potential trading partner is independent of the type of the partner, and is given by the aggregate distribution of money holdings.

[19]The anonymity of each match implies that the current action a household takes in the specialized market will not be known to future households it matches with, except through the level of money holdings the household carries into the match.

[20]Many of the arguments contained below are similar to Lagos and Wright [5].

interest rates. The set of actions and payoffs that are available to the household from period $t+1$ onwards depends on the current actions only through the household's choice of nominal wealth W_{t+1} for next period. In principle, it could also depend on the number of bank notes it brings into next period; however, when bank notes trade at par at the beginning of the next period, as in the equilibria we restrict ourselves to, this cannot be relevant. Let Z_t be the nominal wealth (bank notes, bonds and reserves) that a household has coming into the centralized market. The household problem is

$$\max_{c_{2t}, Y_{2t}} 2\psi\sqrt{c_{2t}} - \eta Y_{2t} + V_{t+1}(W_{t+1})$$

s.t.

$$W_{t+1} = Z_t + P_t(Y_{2t} - c_{2t})$$

and $W_{t+1} \geq 0$. If the solution to this plan involves $c_{2t} > Y_{2t}$, a household submits a bid $P_t(c_{2t} - Y_{2t})$ of nominal wealth in exchange for the credit good; if the reverse inequality applies, a household bids $c_{2t} - Y_{2t}$ of the credit good in exchange for nominal balances.

We first ignore the constraint $Y_{2t} \geq 0$. We then look for parameter values such that this constraint is not binding. V_{t+1} is the value of having a nominal balance of W_{t+1} next period. This value must be well defined in order for the household problem to have a solution (which is necessary for an equilibrium), but at the moment we do not know anything else about it. The solution to this problem yields $c_{2t} = (\psi/\eta)^2$; as for Y_{2t}, two possible cases can arise:

(i) The solution is unbounded; this is incompatible with equilibrium, since, at the aggregate level, market clearing requires $\int x dF_{Y_{2t}}(x) + \epsilon = \int x dF_{c_{2t}}(x)$, where $F_{Y_{2t}}(x)$ ($F_{c_{2t}}(x)$) is the aggregate distribution of output (consumption) plans.

(ii) There is a set of wealth levels $\mathcal{W}_{t+1}$ such that all households, independently of their initial balance Z_t, find it optimal to choose Y_{2t} so that $W_{t+1} \in \mathcal{W}_{t+1}$, and are indifferent among all points in the set. In this case, which must occur in equilibrium, the value of entering the centralized market with a balance of Z_t dollars is $\eta\frac{Z_t}{P_t}$ + constant. We will focus on equilibria in which all households choose the same point for W_{t+1}, independently of their initial wealth Z_t.

Now, we work backwards to look at the problem of a household that is a potential seller of the specialized good in period t. The household faces a trade offer from the buyer $(c_{1t}, \Delta M_t)$, which describes production by the seller and the transfer of money from the buyer. Given the previous discussion, accepting the offer is (weakly) beneficial to the seller if and only if

$$\eta c_{1t} \leq \eta \frac{\Delta M_t}{P_t}$$

A seller will thus accept any offers to trade for which the price is at least P_t, where P_t is the expected prevailing price in the centralized market.

Taking into account what we learned so far, the buyer will offer the seller a price of exactly P_t, and will choose the quantity by solving

$$\max_{c_{1t}} 2\sqrt{c_{1t}} - \eta c_{1t}$$

subject to

$$c_{1t} \leq M_t/P_t$$

The solution to this problem yields an optimal offer of $\min\{M_t/P_t, \eta^{-2}\}$.

Working backwards once again, we consider the problem of a household that has nominal wealth of $W_t + H_t$ and trades with the central bank. If $R_t > 1$, it is clearly optimal to set $X_{t+1} = 0$; if $R_t = 1$, a household is indifferent between bonds and reserves, and we can also set $X_{t+1} = 0$ without loss of generality. If $R_t < 1$, a household can attain infinite nominal wealth by borrowing with bonds and investing in reserves and cash. In this case, it is (at least weakly) optimal to demand an infinite value for $X_{t+1} + B_{t+1}/R_t$, and as many bank notes with infinite denomination as possible.[21] If $R_t \geq 1$, the household problem is

$$\max_{M_t, B_{t+1}} 2\sigma\sqrt{\min\{M_t/P_t, \eta^{-2}\}} - \frac{\eta R_t}{P_t} M_t + (1-\sigma)\frac{\eta}{P_t} M_t$$

If $R_t > 1$, the solution to this problem is

$$c_{1t} = M_t/P_t = \left(\frac{\sigma}{\eta(R_t - 1 + \sigma)}\right)^2 < \frac{1}{\eta^2}$$

[21] We assume that there is free disposal of wealth, so the value must necessarily be weakly increasing with it.

If $R_t = 1$, the solution is $M_t \geq P_t \eta^{-2}$, as cash and bonds become perfect substitutes. When $R_t > 1$, in this stylized model, households need a single bank note from the central bank: they know with certainty the trade that will take place in the specialized market if they meet a seller, and will get a bank note that is of the exact denomination to carry out the transaction. It follows that, as long as bank notes and bonds are expected to trade at par in the future, the constraint on the number of notes that the central bank can issue is not binding.[22] The marginal value of an extra unit of nominal wealth at this stage of the game is therefore $\frac{\eta R_t}{P_t}$. This implies that, as long as $R_t \geq 1$, $V_t'(W_t) = \frac{\beta \eta R_t}{P_t}$ for levels of wealth that are above $\min\{0, -H_t\}$.

Using this information for time $t+1$, we get the following results, as long as $R_{t+1} \geq 1$.

(i) If $\beta R_{t+1} > \pi_{t+1}$, then households would like to produce unbounded amounts of the credit good in period t, which is inconsistent with equilibrium.

(ii) If $\beta R_{t+1} = \pi_{t+1}$, then households are indifferent over their level of the production of the credit good. This corresponds to the main difference equation presented in the text, and is the case we are mainly concerned with. We are interested in equilibria in which all households start from the same level of wealth W_t and attain the same level of wealth W_{t+1}, independently of the type of match they enter into in period t. These equilibria will exist, provided $Y_{2t} \geq 0$ is not binding for any household. Aggregate consumption of the credit good in period t is $(\psi/\eta)^2$. Aggregate voluntary production must thus be $(\psi/\eta)^2 - \epsilon$. In order for W_{t+1} to be the same across households, it must be the case that buyers in the specialized market will compensate the money used there by supplying extra c_{1t} units of the credit good, compared to households that were in a bad match, where c_{1t} was computed above; analogously, sellers must compensate by supplying c_{1t} fewer units. To ensure that sellers can do so without producing negative amounts and independently of the interest rate, we thus need $1/\eta^2 < (\psi/\eta)^2 - \epsilon$, which we assume.

(iii) If $\beta R_{t+1} < \pi_{t+1}$, then either the constraint $W_{t+1} \geq 0$ is binding, or, if $H_{t+1} < 0$ is antici-

[22] When $R_t = 1$, households could again use a single bank note, if it has a denomination of P_t/η^2, or they could use two bank notes, one with the denomination above to trade and the other as an investment, akin to bonds.

pated, $W_{t+1} + H_{t+1} \geq 0$ is binding.[23] The latter constraint cannot arise in an equilibrium, since fiscal policy ensures that $W_{t+1} + H_{t+1}$ is positive in the aggregate. If the former is binding, aggregating across households we obtain

$$P_{t-1}\left[R_t\bar{B} + \pi_t\left(\frac{\sigma}{\eta(R_1 - 1 + \sigma)}\right)^2 (1 - R_t) - \epsilon\pi_t\right] = 0 \tag{13}$$

This equation can be studied with the same techniques of appendix A; in fact, the computations would be identical if $\sigma = 1$. We present here the computations having already taken the limit $\epsilon \to 0$. If the constraint $W_t \geq 0$ were not binding, $R_t = \pi_t$ and the following quadratic equation should be solved:

$$\left(\frac{\pi_t}{\beta} - 1 + \sigma\right)^2 \left(\frac{\bar{B}}{\beta}\right) - \left(\frac{\sigma}{\eta}\right)^2 \frac{\pi_t}{\beta} + \left(\frac{\sigma}{\eta}\right)^2 = 0$$

This equation has no solution when $\beta\sigma - 4\bar{B}\eta^2 < 0$, a weaker condition than the one of appendix A. As before, when $R_t \geq 1$ in all periods, it is not possible for $W_{t+1} \geq 0$ to bind unless $W_t \geq 0$ binds as well. When $W_t \geq 0$ is binding as well, (13) and (6) turn into a difference equation. π_t is a strictly convex function of R_t:

$$\frac{\partial^2 \pi_t}{\partial R_t^2} = 2\bar{B}\eta^2\left[(R_t - 1)^{-3} + \sigma^{-2}\right] > 0$$

It reaches a minimum at

$$\pi_{\min} = \frac{3 + \sqrt{1 + 8\sigma}}{4}$$

with a value of

$$\frac{\bar{B}\,\eta^2\,\left(3 + \sqrt{1 + 8\,\sigma}\right)\,\left(-1 + 4\,\sigma + \sqrt{1 + 8\,\sigma}\right)^2}{16\,\sigma^2\,\left(-1 + \sqrt{1 + 8\,\sigma}\right)}$$

This value is increasing in $\bar{B}$, and decreasing in σ, since its derivative with respect to σ is

$$\frac{-\left(B\,\eta^2\,\left(-1 + \sqrt{1 + 8\,\sigma} + 2\,\sigma\,\left(5 + \sqrt{1 + 8\,\sigma}\right)\right)\right)}{4\,\sigma^3}$$

The conservative value of 600% found in appendix A is thus here even more conservative, since $\sigma < 1$. Repeating the reasoning of appendix A, we look at $\Phi_1(\pi_{\min}; \bar{B}, \sigma)$, where now

[23]The constraint $W_{t+1} + H_{t+1} \geq 0$ comes from the assumption that households are severely punished by the government if their nominal wealth is insufficient to pay taxes in each period.

two parameters need to be set conservatively. We have

$$\frac{\partial^2 \pi_t}{\partial R_t \partial \sigma} = \frac{-2\,\bar{B}\,\eta^2\,\left(-1 + 2\,R_t + \sigma\right)}{\sigma^3} < 0$$

Hence, the value with $\sigma = 1$ from appendix A is a lower bound for the values here, which implies that the solution is indeed explosive. Once again, if the monetary and/or fiscal authorities can rule out explosive inflation paths, this cannot be part of the equilibrium.

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