Online Appendix – Not for Publication

In this part, we provide omitted details of the models and omitted proofs in the main text.

D Proofs and Further Results for the 2-period Model

D.1 Lemma 3

Lemma 3. *1. If $q_1 \geq 1/\phi_1$ an arbitrage is possible;*

2. If $q_1 < 1$, optimal investment by the households is zero;

3. If $1/p_1 < (1-\tau_2^k)r_2/q_1$ an arbitrage is possible;

4. If $1 \leq q_1 \leq 1/\phi_1$ and $1/p_1 > (1-\tau_2^k)r_2(1-\phi_1)/(1-\phi_1 q_1)$, optimal investment by the households is zero;

5. If $q_1 \leq 1/\phi_1$ and $(1-\tau_2^k)r_2/q_1 < 1/p_1 < (1-\tau_2^k)r_2(1-\phi_1)/(1-\phi_1 q_1)$ (implying $q_1 > 1$), optimal sales of claims to capital by the household are strictly positive, but optimal purchases are zero;

6. If $q_1 \leq 1/\phi_1$ and $(1-\tau_2^k)r_2/q_1 < 1/p_1 = (1-\tau_2^k)r_2(1-\phi_1)/(1-\phi_1 q_1)$ (implying $q_1 > 1$), optimal purchases of claims to capital by the household are zero, and households are only willing to undertake investment if they can sell a fraction ϕ_1 to other households.

Proof. In this proof, we label trading strategies as in the main text.

1. Suppose an entrepreneur increases her investment by 1 unit. This comes at a unit resource cost. She can sell a fraction ϕ_1 fetching revenues $\phi_1 q_1 \geq 1$, so the resources available for consumption in period 1 by the household are weakly increased and the constraint (3) is relaxed. Furthermore, the household retains the right to $(1-\phi_1)$ units of capital, which increases resources available for consumption in the second period by $(1-\tau_2^k)r_2(1-\phi_1) > 0$: so each unit of new investment weakly increases consumption in period 1, strictly increases consumption in period 2, and it increases the entrepreneurs' funds available for investment.

2. Trading strategy 2 has a strictly higher return than trading strategies 3 and 4, so investing is strictly dominated by purchasing claims to capital produced by other households. It is therefore optimal to set $k_1^e = 0$.

3. The arbitrage relies on the fact that households are not prevented from shorting government bonds. A household can instruct its workers to purchase claims to capital while returning a negative balance to the household, which the household can in turn cover using borrowed funds (shorting government bonds); each unit of capital costs q_1 and the resulting loan requires a payment of q_1/p_1 in period 2, whereas the purchased capital pays $(1-\tau_2^k)r_2 > q_1/p_1$ in the same period.

4. First note that (conditional on $q_1 < 1/\phi_1$)

$$q_1 > 1 \iff \frac{1}{q_1} < 1 < \frac{1-\phi_1}{1-\phi_1 q_1}:$$

whenever $1 < q_1 < 1/\phi_1$, trading strategy 4 has a strictly higher return than trading strategy 3, which in turn has a strictly higher return than trading strategy 2. In this case, trading strategy 1 has a strictly higher return than strategies 2, 3, and 4. As a consequence, the nonnegativity constraint is binding for the latter three trading strategies, and optimal investment is zero.

5. Trading strategy 1 has a strictly higher return than strategy 2, so optimal purchases of claims to capital by the household are zero. Trading strategy 4 has a strictly higher return than strategy 1, so optimal investment is positive and as high as permitted by constraints (2) and (3). Finally, since $q_1 > 1$, trading strategy 4 has a strictly higher return than trading strategy 3, so the household finds it optimal to sell as much capital as allowed by (2): with strictly positive investment, this implies strictly positive sales are optimal.

6. This case is very similar to the previous one, except that trading strategies 1 and 4 have the same rate of return. A household is indifferent between undertaking levered investment or investing in government bonds; however, it strictly prefers buying government bonds to investing unless investment is financed by outside funds as much as allowed by (2), and it strictly prefers buying government bonds to purchasing claims to capital. Optimal purchases of capital are zero; either investment is zero, or, if it strictly positive, then optimal sales of claims to capital are strictly positive as well.

□

Collecting all of the cases that are ruled out by Lemma 3, the set of prices, taxes, and interest rates that are left are those described by (11) and (12) in the main text.

D.2 Proof of Proposition 1

Suppose that the vector $(C_1, C_2, L_1, L_2, K_1)$ satisfies (9), (10), and (16). This allocation is optimal for the firms if w_1, w_2, and r_2 are set according to (14). Substituting factor prices and $L_t = (1-\chi)\ell_t$ into equation (13), this equation can be made to hold for a suitable choice of τ_t^ℓ.[42] We set $p_1 = \beta$, as we proved that this is necessary for an equilibrium, and τ_2^k so that (11) holds. With these choices, a household is indifferent on the timing of consumption between periods 1 and 2, as long as its budget constraint is exhausted.

Next, we proceed separately for the two cases: $K_1 \leq K^*$ and $K_1 > K^*$.

Suppose first that $K_1 \leq K^*$. If we set $q_1 = 1$, households are indifferent on their investment level, so K_1 is weakly optimal, provided it satisfies (2) and (3). With $q_1 = 1$, any choice of sales and purchases of claims to capital is also weakly optimal, as long as they satisfy the same equations. One possible solution is $s_1^e = \max\{0, k_1^e - b_0^e\} = S_1^e/\chi = S_1^w/\chi = s_1^w(1-\chi)/\chi$, which makes (3) hold with equality and implies that (2) holds as well. This solution satisfies market clearing for claims to capital. Generically, the solution is not unique, since households are indifferent at the margin between selling capital, buying capital, or investing their own funds in capital produced by their entrepreneurs.[43]

Finally, we need to verify that the budget constraints of the households or those of the government are satisfied (Walras' law implies that the government budget constraints are satisfied if those of the

[42]Note that both sides of the equation are positive, so the solution implies $\tau_t^\ell < 1$; it is possible that it features $\tau_t^\ell < 0$, which corresponds to labor subsidization.

[43]$q_1 > 1$ is impossible in this case, as long as $K_1 > 0$: with $q_1 > 1$, households would optimally sell a fraction ϕ_1 of the capital that they produce, but not buy any of the capital produced by the entrepreneurs of other households, so market clearing would be impossible.

households are, and vice versa). We substitute the allocation and the prices and taxes that we derived above into the household budget constraint in period 1 and we obtain

$$C_1 = v'(L_1)L_1 + B_0^w - \beta B_1 + B_0^e - K_1. \tag{66}$$

We solve equation (66) for B_1, thereby ensuring that it holds. Substituting this value of B_1 along with prices and taxes into (6) we obtain the first case of (16), thereby verifying that the budget constraints hold at the given prices.

Second, suppose that $K_1 > K_1^*$. Set $S_1^e = \phi_1 K_1$ and $q_1 = (K_1 - B_0^e)/(\phi_1 K_1)$, so that (2) and (3) hold with equality. When $K_1 > K^*$, the resulting value for q_1 is strictly greater than 1, so the household finds it optimal to invest and sell as much of the capital produced by its entrepreneurs as possible, which is consistent with (2) and (3) binding. Market clearing requires $S_1^w = S_1^e$; this choice is weakly optimal for the household given (11).[44] Repeating the steps for the case $K_1 \leq K^*$ we compute prices and taxes, and we substitute them into the household budget constraint in period 1, obtaining (66) again. This can solved for B_1 as in the case of $K_1 < K^*$. Substitution of the resulting value of B_1 into (6) yields the second case of (16). This concludes the proof that any vector that satisfies (9), (10), and (16) is part of a competitive equilibrium.

To proceed in reverse, any allocation that does not satisfy (9) or (10) is not part of a competitive equilibrium, since those conditions are necessary. Consider any allocation that does not satisfy (16). We can repeat the steps that we used before to deduce prices, taxes, and B_1 from the necessary conditions for a competitive equilibrium, and substitute them into the budget constraint of the households. If (16) fails, then at the given prices, taxes, and B_1, the budget constraint (6) will also fail. Specifically, if the left-hand side of (16) is larger than the right-hand side, the resulting allocation, prices, and taxes, violate the household budget constraint. If instead the left-hand side is smaller, they violate the government budget constraint.

E Endogenous Asset Liquidity: a Microfoundation

The intermediation technology follows mostly Cui and Radde (2020) and Cui (2016). There are capital submarkets, denoted by superscripts $m = 1, 2, 3, ...$. As we shall see, the number of submarkets is not important. On each submarket, entrepreneurs and workers post U^m units of sell orders and V^m units of buy orders, respectively. If matched, intermediaries ensure that buyers have enough resources to fill buy orders; sell orders U^m need to be backed by private claims, i.e., each entrepreneur cannot post more than the sum of new and old capital for sale.[45]

There is a continuum of competitive financial intermediaries. Each chooses on which submarket to collect and match quotes at per-quote costs of κ units of consumption goods. The probability of filling a buy quote is f^m, while the probability of filling a sell order (or asset saleability) is ϕ^m.

On each submarket m, financial intermediaries' gross profit amounts to the difference between the competitive buy price $q^{w,m}$ collected from workers and the sell price q^m paid to entrepreneurs on the fraction of successfully matched quotes. Notice that workers direct their quotes to the submarket with the lowest purchase price $q^{w,m} = q^w$, which is taken as given by intermediaries.

[44] In this case, setting $q_1 = 1$ would not be compatible with an equilibrium, since either (2) or (3) would be violated by any choice of S_1^e.

[45] This assumption is for the existence of binding financing constraints. If entrepreneurs can freely post sale orders, they will post the number of orders (give the probability of matching ϕ) to undo financing constraints. We could relax the assumption and allow entrepreneurs to post a fraction $x > 1$ of new and used capital, as long as x is not too large.

Since financial intermediaries operate in a competitive environment, they earn zero (net) profit from each transaction, i.e., $\kappa/f^m = q^{w,m} - q^m$. In light of this zero-profit condition, intermediaries are indifferent between all submarkets and we can omit the superscript m:

$$\frac{\kappa}{f} = q^w - q \tag{67}$$

The corresponding $\eta(\phi)$ function in the main text is the same as κ/f, an increasing function of ϕ.

The matching probabilities depend on intermediaries' matching technology. This technology is characterized by a matching function

$$M(U,V) = \xi U^{\gamma} V^{1-\gamma}$$

where ξ captures matching efficiency and γ is the matching elasticity with respect to sell orders U. Then, asset saleability and the probability of filling buy orders are

$$\phi \equiv \frac{M(U,V)}{U}, \quad f \equiv \frac{M(U,V)}{V} = \xi^{\frac{1}{1-\gamma}} \phi^{\frac{\gamma}{\gamma-1}} \tag{68}$$

Defining market tightness θ as the ratio of buy orders to sell orders, that is, $\theta \equiv V/U$, asset liquidity ϕ has a one-to-one mapping relationship with θ.

Entrepreneurs post orders amounting to $U = K^e + (1-\delta)\chi K_{-1}$, of which a fraction $\phi U = M$ is sold. In this sense, ϕ indeed captures asset saleability. Their optimal choice of which market to choose for their sales is dictated by Lemma 1. In equilibrium, financial intermediaries operate only in the market that minimizes (30) subject to (31) and (32), since the price in other markets would not attract any entrepreneurs (or would not allow intermediaries to break even). Similarly, workers post total orders $V = f^{-1}[S^w]$ and they have enough resources to fill matched buy orders (as they are not financing constrained).

If we set $\gamma = 1/2$, then κ/f in (67) becomes $\kappa\xi^{-2}\phi^2$. Therefore, the cost function $\eta(\phi) = \omega_0\phi^{\omega_1}$ used in the main text can be obtained if we set $\omega_0 = \kappa\xi^{-2}$ and $\omega_1 = 2$.

F Proofs and Other Results for Section 3

F.1 Proof of Lemma 1

- By contradiction, suppose that (k_t^e, s_t^e, ϕ_t) do not solve the given problem. Let $(\tilde{k}, \tilde{s}, \tilde{\phi})$ be an alternative triple that achieves a strictly lower cost while respecting the constraints (31) and (32). If we replace (k_t^e, s_t^e, ϕ_t) by $(\tilde{k}, \tilde{s}, \tilde{\phi})$ into (27), equation (32) implies that capital accumulation by the household is no smaller than before. The second inequality in (24) holds for the alternative allocation since it is precisely (31), and the first inequality is strictly relaxed, since the contradiction assumption implies $q_t(\phi_t)s_t^e - k_t^e > \hat{q}(\hat{\phi})\hat{s} - \hat{k}$. This in turn implies that the budget constraint (26) is also strictly relaxed, since entrepreneurs bring more resources for consumption at the end of the period, and the household could improve upon its allocation by increasing period-t consumption without ever having to increase consumption or the labor supply in any other period; this would then imply that (k_t^e, s_t^e, ϕ_t) is not optimal.

- From (20), q is a strictly decreasing (and concave) function of ϕ. This implies that (31) must hold as an equality at the optimum, for otherwise the household could choose a lower value of $\hat{\phi}$ and the same value for k_t^e and s_t^e and still satisfy (31) and (32), while lowering the cost of this investment (which implies increasing resources available for consumption).[46] After using (31) to substitute out s_t^e, the problem becomes

$$\min_{(\hat{k},\hat{\phi})}[\hat{k} + (1-\delta)k_{t-1}][1 - \hat{\phi}q_t(\hat{\phi})] - (1-\delta)k_{t-1}$$

subject to

$$[\hat{k} + (1-\delta)k_{t-1}][1-\hat{\phi}] \geq k_t^e + (1-\delta)k_{t-1} - s_t^e. \tag{69}$$

We now see that (69) must also be binding, and use it to substitute for $\hat{k}$, obtaining

$$\min_{\hat{\phi}}[k_t^e + (1-\delta)k_{t-1} - s_t^e]\frac{1-\hat{\phi}q_t(\hat{\phi})}{1-\hat{\phi}} + (1-\delta)k_{t-1},$$

completing the proof.

F.2 Proof of Lemma 2

- If $q_t^w < 1$, we proved that the optimal choice of ϕ_t is zero (and thus $q_t^* = q_t^w$), so no claims to capital are sold. If the optimal household choice is $s_t^w > 0$, the given prices and taxes cannot form part of an equilibrium, since market clearing requires $S_t^w = S_t^e$. If instead the optimal choice is s_t^w, suppose we now raise the price q_t^w to 1, which implies q_t^* also is raised to 1. On the selling side, $\phi_t = 0$ remains optimal and thus so is $s_t^e = 0$. The choice of investment k_t^e is unaffected by q_t when $\phi_t = 0$. $s_t^w = 0$ is a fortiori optimal at the new higher price (and the constraint $s_t^w \geq 0$ must be binding at the new price). Hence, the same allocation remains optimal for the household. This price change has no effect on the firm or government problem, and intermediaries still break even at the new price schedule implied by (20) with $q_t^w = 1$.

- As in the previous point, we have $\phi_t = 0$ and hence $s_t^e = 0$. Looking at the household budget constraint (28) and capital evolution equation (29), a unit increase in χk_t^e or a unit increase in $(1-\chi)s_t^w$ decrease resources available for consumption in period t by the same amount, and increase capital holdings in period $t+1$ (k_t) also by the same amount.

- If $q_t^w > 1$, we have $\phi_t > 0$, $q_t^* < q_t^w$, and from Lemma 1 equation (31) holds as an equality at an optimal choice by the household, which implies $s_t^e > 0$. Substituting this equation, the household budget constraint becomes

  $$c_t + p_t b_t + \chi k_t^e(1 - q_t^*\phi_t) = (1-\tau_t^\ell)w_t(1-\chi)\ell_t + b_{t-1} + [r_t(1-\tau_t^k) + \delta\tau_t^k + \chi\phi_t q_t^*(1-\delta)]k_{t-1} - (1-\chi)q_t^w s_t^w$$

  the capital evolution equation is

  $$k_t = (1-\delta)k_{t-1} + (1-\chi)s_t^w + \chi(1-\phi_t)k_t^e - \chi\phi_t(1-\delta)k_{t-1},$$

[46] Once ϕ hits zero, the household cannot lower it any further, but at that point it must also be that $s_t^e = 0$ and (31) has to hold as an equality nonetheless.

and the financial constraint is

$$k_t^e(1-\phi_t q_t^*) \le b_{t-1} + \phi_t q_t^*(1-\delta)k_{t-1}. \tag{70}$$

If $s_t^w = 0$ is optimal, then at the given prices and taxes the household finds it optimal to sell some claims to capital (possibly just undepreciated capital from the previous period), but not to buy any. Suppose instead by contradiction that $s_t^w > 0$ is optimal and that (70) does not bind. Then the household could consider the following perturbation: decrease s_t^w by $\epsilon/(1-\chi)$ and increase k_t^e by $\epsilon/[\chi(1-\phi_t)]$. This perturbation leaves capital k_t unaffected, respects (70) for ϵ sufficiently small, and increases resources available for consumption in period t by

$$-\frac{1-\phi_t q_t^*}{1-\phi_t} + q_t^w = \frac{q_t^w - 1 - \phi_t(q_t^w - q_t^*)}{1-\phi_t} > 0,$$

thereby contradicting the assumption that the original allocation is optimal.

F.3 Proof of Proposition 3

Proof. Consider first an allocation that satisfies the conditions in the proposition. We can infer the equilibrium value of K_t^e from equation (29). From Lemma 1, the values of S_t^e must satisfy

$$S_t^e = \phi_t[K_t - (1-\chi)(1-\delta)K_{t-1}]. \tag{71}$$

and q_t^w is set according to equation (34). The schedule $q_t(\phi)$ is then given by (20). Market clearing requires $S_t^w = S_t^e$. Factor prices w_t and r_t are pinned down by the firms' optimality conditions (21). We can infer the tax rate on labor from (38), and the price of bonds from (39) and (37), where $q_t^* = q_t(\phi_t)$ as defined in the main text. We recover $B_t = \tilde{B}_t/p_t$, and the tax rate on capital (except for the exogenously given τ_0^k) from equation (40). This constraint is equivalent to (44) when expressed in terms of aggregate variables. Finally, after we substitute the appropriate values of p_t, B_t, q_t^w, ρ_t (as defined in (37)), $q_t^* = q_t(\phi_t)$, r_t, τ_t^k, τ_t^ℓ, and w_t, equations (36) and (42) are equivalent for period $t > 0$, and so are equations (36) and (43) for time 0, (44) and (70) at $t > 0$, and (45) and (70) at $t = 0$.

Conversely, suppose that an allocation does not satisfy the conditions in the proposition. From necessary conditions for a competitive equilibrium we could derive $K_t^e, S_t^e, S_t^w, q_t^w, q_t(.), w_t, r_t, \tau_t^\ell, \tau_t^k, p_t, B_t$, and τ_t^k as above. If (42) does not hold for some period t, then the same substitutions as in the previous step (but in reverse) would imply that can (36) does not hold either. Similarly, if (43) fails, then (36) fails at time 0. If (44) fails, then so does (70); for time 0, if (45) fails, then so does (70). In all of these cases, the allocation would not be part of a competitive equilibrium. □

F.4 Relationship between the relaxed and the original Ramsey problem

In the characterization of the set of competitive-equilibrium allocations, equation (44) has to hold as an equality if $\phi_t > 0$. We show here that any interior solution to the problem remains the same if we impose it as a weak inequality for all values of $\phi_t \in [0,1]$. Suppose that the solution of maximizing (46) subject to (41), (42), (43), and subject to (44) treated as a weak inequality even for $\phi_t > 0$ is interior, and suppose that the financial constraint (44) is not binding; Letting $\beta^t\lambda_t$ be the Lagrange multiplier on constraint (41), the first-order effect of ϕ_t (with (44) slack) on the Lagrangean for $t > 0$

is given by

$$-\lambda_t[K_t-(1-\chi)(1-\delta)K_{t-1}][\eta_t+\phi_t\eta_t']+(q_t^w)'(\Psi_t-\Psi_{t+1})u'(C_t)K_t.$$

The first-order condition for $\tilde{B}_t$ implies $\Psi_{t+1}\geq\Psi_t$, so the expression above is strictly negative for all $\phi_t\in(0,1)$. The proof for time 0 is similar; in this case, the first-order effect is

$$\begin{aligned}&-\lambda_0[K_0-(1-\chi)(1-\delta)K_{-1}][\eta_0+\phi_0\eta_0']\\&+u'(C_0)\left[(q_0^w)'(\Psi_0-\Psi_1)K_0-\Psi_0(q_0^w)'(1-\delta)K_{-1}\left(1+\frac{\chi\phi_0(q_0^*-1)}{1-\phi_0q_0^*}\right)\right]\\&-u'(C_0)\left[\Psi_0\left(q_0^*-1+(1-\phi)(q_0^*)'\right)\frac{(1-\delta)\chi K_{-1}q_0^w}{(1-\phi_0q_0^*)^2}+\Psi_0B_{-1}\chi\rho'\right],\end{aligned}$$

which is also strictly negative for all $\phi_0\in(0,1)$.[47] This proves that the relaxed planner's problem always features $\phi_t=0$ whenever the financial constraint is not binding in period t, implying then that the constraint binds whenever the relaxed problem has $\phi_t>0$: the solution of the relaxed problem thus coincides with the Ramsey plan.

F.5 First-order conditions of the Infinite-Horizon Planner's Problem at $t=0$

- consumption in period 0:

$$\begin{aligned}&(1+\Psi_0)u'(C_0)+\Psi_0u''(C_0)[C_0+\tilde{B}_0+q_0^wK_0]\\&-\Psi_0u''(C_0)\left\{B_{-1}(1+\chi\rho_0)+\left[(1-\tau_0^k)F_K(K_{-1},L_0)+\delta\tau_0^k+q_0^w(1-\delta)\left(1+\frac{\chi\phi_0(q_0^*-1)}{1-\phi_0q_0^*}\right)\right]K_{-1}\right\}\\&-\lambda_0=-\gamma_1u''(C_0)\frac{\chi\tilde{B}_0}{\beta(1+\chi\rho_1)};\end{aligned}$$

- leisure in period 0:

$$v'(L_0)(1+\Psi_0)+\Psi_0v''(L_0)L_0-\lambda_0F_L(K_{-1},L_0)+\Psi_0u'(C_0)(1-\tau_0^k)F_{KL}(K_{-1},L_0)K_{-1}=0;$$

- liquidity in period 0:

$$\begin{aligned}&-\lambda_0[K_0-(1-\chi)(1-\delta)K_{-1}][\eta_0+\phi_0\eta_0']\\&+u'(C_0)\left[(q_0^w)'(\Psi_0-\Psi_1)K_0-\Psi_0(q_0^w)'(1-\delta)K_{-1}\left(1+\frac{\chi\phi_0(q_0^*-1)}{1-\phi_0q_0^*}\right)\right]\\&-u'(C_0)\left[\Psi_0\left[q_0^*-1+\phi_0(1-\phi_0)(q_0^*)'\right]\frac{(1-\delta)\chi K_{-1}q_0^w}{(1-\phi_0q_0^*)^2}-\Psi_0B_{-1}\chi\rho_0'\right]\\&+\gamma_0[K_0-(1-\chi)(1-\delta)K_{-1}]\left[q_0^*+\phi_0\left(q_0^*\right)'\right]=0.\end{aligned}$$

[47] To get positive signs for the Lagrange multipliers, the right-hand side of the resource constraint (41) must be weakly larger than the left-hand side, reflecting the fact that the social value of extra production is positive, and the left-hand side of the implementability constraints (42) and (43) must be weakly bigger than the right-hand side, which is the way these constraints would appear if we allowed the planner to use lump-sum transfers but not lump-sum taxes.

- capital in period 0:

$$
\begin{aligned}
&\lambda_0 \left(1+\phi_0\eta_0\right) - \Psi_0 u'(C_0) q_0^w + \gamma_0(1-\phi_0 q_0^*) \\
=&\beta\lambda_1 \left\{F_K(K_0, L_1) + \left[1+(1-\chi)\phi_1\eta_1\right](1-\delta)\right\} - \Psi_1 u'(C_0) q_0^w \\
+&\beta\gamma_1 u'(C_1)\left[1-(1-\chi)\phi_1 q_1^*\right](1-\delta);
\end{aligned}
$$